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# Polynomial Approximation of an Entire Function of Slow Growth

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## 1. INTRODUCTION

Let $C[-1, 1]$ be the set of all real or complex valued continuous functions defined on the closed interval $[-1, 1]$. If $f(x) \in C[-1, 1]$, let

$$E_n(f) = \inf_{p \in \pi_n} \|f - p\|, \qquad n = 0, 1, 2, \ldots, \tag{1.1}$$

where the norm is the sup norm on $[-1, 1]$ and $\pi_n$ denotes the set of all polynomials $p$ of degree at most $n$. Bernstein ([1, p. 118]; see also [5, pp. 76–78; 6, pp. 90–94]) proved that

$$\lim_{n\to\infty} (E_n(f))^{1/n} = 0 \tag{1.2}$$

if and only if $f(x)$ is the restriction to $[-1, 1]$ of an entire function. Varga [13] studied the order and type of this entire function. Reddy [7, 8] further studied different orders and different types of $f(z)$ and Juneja [3] studied its lower order.

Let $L^0$ denote the class of functions $h(x)$ satisfying conditions (H, i) and (H, ii):

(H, i) $h(x)$ is defined on $[a, \infty)$; is positive, strictly increasing, and differentiable; and tends to $\infty$ as $x \to \infty$.

(H, ii) $$\lim_{x\to\infty} \frac{h[x(1 + \tilde{g}(x))]}{h(x)} = 1$$

for every function $\tilde{g}(x)$ such that $\tilde{g}(x) \to 0$ as $x \to \infty$.



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Let $\Lambda$ denote the class of functions $h(x)$ satisfying conditions (H, i) and (H, iii):

$$\text{(H, iii)} \qquad \lim_{x\to\infty} \frac{h(cx)}{h(x)} = 1 \qquad \text{for every } c, \quad 0 < c < \infty.$$

Following Šeremeta [9], Shah [10] defined generalised order $\rho(\alpha, \beta, f)$ and generalised lower order $\lambda(\alpha, \beta, f)$ of an entire function $f(z)$ as

$$\begin{matrix} \rho(\alpha,\beta,f) \\ \lambda(\alpha,\beta,f) \end{matrix} = \lim_{r\to\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\alpha(\log M(r,f))}{\beta(\log r)}, \tag{1.3}$$

where $\alpha(x) \in \Lambda$ and $\beta(x) \in L^0$, and generalised various results contained, for example, in [3, 7, 8, 13].

Taking $\alpha(x) = \log x$ and $\beta(x) = x$ in (1.3) we get the familiar order $\rho \equiv \rho(f)$ and the lower order $\lambda \equiv \lambda(f)$ of an entire function $f(z)$ [2, p. 8]. An entire function $f(z)$ for which $\rho = 0$ is said to be slow growth. Various authors (e.g., [4, 11]) have defined order and lower order of an entire function $f(z)$ of slow growth by considering the ratio $l_j M(r,f)/l_j r$, $j \geqslant 2$, where $l_1 x = \log x$, $l_j x = \log(l_{j-1} x)$.

The generalised orders of an entire function in terms of the coefficients in its Taylor series are characterized by Shah [10]. Some of his results [10, (1.6), (1.7)] are obtained under the condition

$$\frac{d[\beta^{-1}(\alpha(x))]}{d(\log x)} = O(1) \qquad \text{as} \quad x \to \infty. \tag{1.4}$$

Clearly (1.4) is not satisfied for $\alpha(x) = \beta(x)$. Thus, in this case, the corresponding resuls of Shah [10, (1.6), (1.7)] are not applicable.

In the present paper we define generalised orders of an entire function in a new way. Our results apply satisfactorily to entire functions of slow growth and generalise many previous results [4, 7, 8, 11].

Let $\Omega$ be the class of functions $h(x)$ satisfying (H, i) and (H, iv):

(H, iv) There exists a $\delta(x) \in \Lambda$ and $x_0$, $K_1$ and $K_2$ such that

$$0 < K_1 \leqslant \frac{d(h(x))}{d(\delta(\log x))} \leqslant K_2 < \infty$$

for all $x > x_0$.

Let $\bar{\Omega}$ be the class of functions $h(x)$ satisfying (H, i) and (H, v):

$$\text{(H, v)} \qquad \lim_{x\to\infty} \frac{d(h(x))}{d(\log x)} = K, \qquad 0 < K < \infty.$$

It can easily be seen that $\Omega$ and $\bar{\Omega}$ are contained in $\Lambda$. Further $\Omega$ and $\bar{\Omega}$ have no common element. Let $F_{q,j}(x) = \exp_q(cl_{j+1}x)$, where $\exp_0(w) = w$ and $\exp_q(w) = \exp(\exp_{q-1} w)$, $q = 1, 2, \ldots$. Then, the functions $F_{0,p}$, $F_{2,p+1}$, with $0 < c < 1$ if $p = 1$ and $0 < c < \infty$ if $p > 1$, and $F_{p,p}$, $p \geqslant 2$, $0 < c < 1$, are in $\Omega$ with the choices of $\delta(x)$ as $F_{0,p-1}$, $F_{2,p}$ and $F_{p,p-1}$, respectively. In fact all the functions of the form $\delta(\log x)$, where $\delta(x) \in \Lambda$, are in $\Omega$. The functions $(\alpha_1 - \alpha_2/x) \log x$, $\alpha_1 > 0$, $\alpha_2 \geqslant 0$, and $\log x + \alpha_3 (l_p x)^{\alpha_4}$, where $0 < \alpha_3 < \infty$ and $\alpha_4 \leqslant 1$ if $p = 1$ and $0 < \alpha_4 < \infty$ if $p > 1$, are in $\bar{\Omega}$. Since $h(x) \in \Omega$ implies $h(x) \sim h(x^2)$ as $x \to \infty$, the functions $F_{1,1}$ with $0 < c < 1$ and $h_1(x) \log x$, where $h_1(x)$ satisfies (H, i) are neither in $\Omega$ nor in $\bar{\Omega}$.

Let

$$f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n} \tag{1.5}$$

be a nonconstant entire function. Here $\lambda_0 = 0$ and $\{\lambda_n\}_{n=1}^{\infty}$ is a strictly increasing sequence of positive integers such that no element of the sequence $\{a_n\}_{n=1}^{\infty}$ is zero.

We define the generalised order $\rho(\alpha, \alpha, f)$ and the generalised lower order $\lambda(\alpha, \alpha, f)$ of the entire function $f(z)$, given by (1.5), as

$$\begin{matrix} \rho(\alpha, \alpha, f) \\ \lambda(\alpha, \alpha, f) \end{matrix} = \lim_{r \to \infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\alpha(\log M(r, f))}{\alpha(\log r)}, \tag{1.6}$$

where $\alpha(x)$ either belongs to $\Omega$ or to $\bar{\Omega}$ and

$$M(r) \equiv M(r, f) = \max_{|z| = r} |f(z)|.$$

We remark here that if $\alpha(x) \in \bar{\Omega}$ then generalized orders of $f(z)$, given by (1.6), coincide with its (2, 2) orders [4]. There are certain functions (e.g., $(\log x)^c$, $0 < c < 1$, or $(\log x)^{\alpha_1} (l_2 x)^{\alpha_2} \cdots (l_p x)^{\alpha_p}$, where $\alpha_1 \geqslant 1$ and at least one $\alpha_i$, $i = 2, \ldots, p$, is nonzero if $\alpha_1 = 1$) which are inadmissible in $\Omega$ or $\bar{\Omega}$ but if the rate of growth of an entire function with respect to such functions is measured by (1.6) with $1 < \rho(\alpha, \alpha, f) < \infty$, then the same is as well measured by functions in $\bar{\Omega}$. Thus, excluding these functions from the classes $\Omega$ or $\bar{\Omega}$ does not mean excluding entire functions of certain types of growth from our discussion.

Further, let

$$\mu(r) \equiv \mu(r, f) = \max_{n \geqslant 0} \{|a_n| \, r^{\lambda_n}\}$$

and

$$\nu(r) \equiv \nu(r, f) = \max\{\lambda_n : \mu(r) = |a_r| \, r^{\lambda_n}\}.$$

The functions $\mu(r)$ and $\nu(r)$ are called, respectively, the maximum term and the rank of the maximum term of $f(z)$ for $|z| = r$.

In Theorem 3 we obtain $\rho(\alpha, \alpha, f)$ and $\lambda(\alpha, \alpha, f)$ in terms of $\mu(r)$ and $\nu(r)$. Theorems 4–6 deal with characterizations of generalized orders in terms of $a_n$'s. We then apply these results in Theorems 1 and 2 to obtain expressions for generalised orders in terms of the approximation error $E_n(f)$.

We shall use the following notations throughout the paper.

*Notation* 1.

$$\begin{aligned} P_{\varphi}(\xi) &= \max\{1, \xi\} \qquad &\text{if} \quad \alpha(x) \in \Omega \\ &= \varphi + \xi \qquad &\text{if} \quad \alpha(x) \in \bar{\Omega}. \end{aligned}$$

We shall write $P(\xi)$ for $P_1(\xi)$.

*Notation* 2. $G[x; c] = \alpha^{-1}[c\alpha(x)]$, $c$ is a positive constant.

*Notation* 3. $\psi(n)$, $n = 0, 1, 2, \ldots$, will denote the function

$$\psi(n) = \frac{1}{\lambda_{n+1} - \lambda_n} \log \left| \frac{a_n}{a_{n+1}} \right|.$$

## 2. Main Theorems

Throughout Sections 2 and 4 we shall assume that $f(x) \in C[-1, 1]$ is not a polynomial.

Theorem 1. *Let $f(x) \in C[-1, 1]$ and $E_n(f)$, defined by* (1.1), *satisfy* (1.2). *Then,*

(A) *$f(x)$ is restriction to $[-1, 1]$ of an entire function $f(z)$.*

*Also*

(B) (i) *$\rho(\alpha, \alpha, f) = P(L)$, where*

$$L = \limsup_{n\to\infty} \frac{\alpha(n)}{\alpha((1/n)\log(1/E_n(f)))}.$$

(ii) *$\rho(\alpha, \alpha, f) \leqslant P(L^*)$ if $L^*$ is well defined by*

$$L^* = \limsup_{n\to\infty} \frac{\alpha(n)}{\alpha\{\log(E_{n-1}(f)/E_n(f))\}}.$$

(iii) *$\lambda(\alpha, \alpha, f) \geqslant P(\hat{l})$, where*

$$\hat{l} = \liminf_{n\to\infty} \frac{\alpha(n)}{\alpha\{(1/n)\log(1/E_n(f))\}}.$$

(iv) *If we take* $\alpha(x)=\alpha(a)$ *on* $(-\infty, a)$ *then*

$$\lambda(\alpha, \alpha, f) \geqslant P(l^*),$$

*where*

$$l^* = \liminf_{n\to\infty} \frac{\alpha(n)}{\alpha\{\log(E_{n-1}(f)/E_n(f))\}}.$$

(C) *Further, if* $E_n(f)/E_{n+1}(f)$ *is ultimately nondecreasing, then*

$$\rho(\alpha, \alpha, f) = P(L) = P(L^*)$$

*and*

$$\lambda(\alpha, \alpha, f) = P(l) = P(l^*).$$

THEOREM 2. *Let* $f(x) \in C[-1, 1]$ *and* $E_n(f)$ *satisfy* (1.2). *Then*

(i) *If* $\alpha(x) \in \Omega$, *we have*

$$\lambda(\alpha, \alpha, f) = \max_{\{n_k\}}\{P_{\chi_t}\{l_t\}\} \tag{2.1}$$

*and if we further take* $\alpha(x)=\alpha(a)$ *on* $(-\infty, a)$, *then*

$$\lambda(\alpha, \alpha, f) = \max_{\{n_k\}}\{P_{\chi_t}\{l_t^*\}\}, \tag{2.2}$$

*where*

$$\chi_t \equiv \chi_t(\{n_k\}) = \liminf_{k\to\infty} \frac{\alpha(n_{k-1})}{\alpha(n_k)}$$

*and*

$$l_t \equiv l_t(\{n_k\}) = \liminf_{k\to\infty} \frac{\alpha(n_{k-1})}{\alpha\{(1/n_k)\log(1/E_{n_k}(f)\}}$$

*and*

$$l_t^* \equiv l_t^*(\{n_k\}) = \liminf_{k\to\infty} \frac{\alpha(n_{k-1})}{\alpha\{(1/(n_k - n_{k-1}))\log(E_{n_{k-1}}(f)/E_{n_k}(f))\}}.$$

*Maximum in* (2.1) *and* (2.2) *is taken over all increasing sequences* $\{n_k\}$ *of positive integers.*

(ii) *Further, if* $\{n_m\}$ *is the sequence of the principal indices of the entire function* $g(z) = \sum_{n=0}^{\infty} E_n(f)\, z^n$ *and* $\alpha(n_m) \sim \alpha(n_{m+1})$ *as* $m \to \infty$, *then* (2.1) *and* (2.2) *also hold for* $\alpha(x) \in \bar{\Omega}$. *(Here again we taken* $\alpha(x) = \alpha(a)$ *on* $(-\infty, a)$.)

*Remark.* With $\alpha(x) = \log x$ in Theorem 1, some results of Reddy [7, 8] follow.

## 3. Some Intermediary Theorems

Throughout this section we shall assume that the entire function $f(z)$, defined by (1.5), is not a polynomial.

THEOREM 3. *Let* $f(z)$ *be an entire function defined by* (1.5). *Then*

$$\rho(\alpha, \alpha, f) = P(\varphi_1) = \theta_1 \tag{3.1}$$

*and*

$$\lambda(\alpha, \alpha, f) = P(\varphi_2) = \theta_2, \tag{3.2}$$

*where*

$$\begin{matrix} \varphi_1 \\ \varphi_2 \end{matrix} = \lim_{r \to \infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\alpha(\nu(r))}{\alpha(\log r)} \tag{3.3}$$

*and*

$$\begin{matrix} \theta_1 \\ \theta_2 \end{matrix} = \lim_{r \to \infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\alpha(\log \mu(r))}{\alpha(\log r)}. \tag{3.4}$$

*Proof.* We shall prove the theorem in several parts.

(i) Since $\log M(r)$ is a convex function of $\log r$ we have $\rho(\alpha, \alpha, f) \geqslant \lambda(\alpha, \alpha, f) \geqslant 1$.

(ii) $\rho(\alpha, \alpha, f) = \theta_1$ and $\lambda(\alpha, \alpha, f) = \theta_2$ follow easily on the lines of proof of Theorem 1 of Shah [10].

(iii) Let $\alpha(x) \in \Omega$. Since [2, pp. 12, 13; 12, pp. 28–32]

$$\log \mu(er) > \nu(r),$$

using parts (i) and (ii) of the proof, we have $\theta_2 \geqslant \max(1, \varphi_2)$.

To prove $\theta_2 \leqslant \max(1, \varphi_2)$ assume that $\varphi_2 < \infty$. First let $1 \leqslant \varphi_2 < d < \infty$. Then, from (3.3), we have

$$\nu(r_n) < G[\log r_n; d] \tag{3.5}$$

for a sequence $\{r_n\}$, $r_n \to \infty$. From (3.5) and [2, pp. 12, 13] we get

$$(1 + o(1)) \log \mu(r_n) < \nu(r_n) \log r_n \leqslant \{G[\log r_n; d]\}^2$$

for $\{r_n\}$, since $d > 1$. This gives $\theta_2 \leqslant \varphi_2$, if $\varphi_2 \geqslant 1$, since $\alpha(x) \in \Omega$. Now, let $\varphi_2 < 1$. Then, from (3.3)

$$\nu(r'_n) < \log r'_n \tag{3.6}$$

for a sequence $\{r'_n\}$. Using (3.6) and the fact that $\alpha(x) \in \Omega$ one gets $\theta_2 \leqslant 1$ and so $\theta_2 = \max(1, \varphi_2)$.

(iv) Proof of $\theta_1 = \max(1, \varphi_1)$ when $\alpha(x) \in \Omega$ is similar to part (iii) above.

(v) Now, let $\alpha(x) \in \bar{\Omega}$. We have [12, pp. 28–32]

$$\alpha\{(1 + o(1)) \log \mu(r)\} < \alpha(\nu(r)) + \log \log r \left. \frac{d\alpha(x)}{d \log x} \right|_{x = x^*(r)},$$

where $\nu(r) < x^*(r) < \nu(r) \log r$. This gives $\theta_2 \leqslant 1 + \varphi_2$, since $\alpha(x) \in \bar{\Omega}$.

Since [12, pp. 28–32] $\log \mu(r^2) > \nu(r) \log r$, proceeding as above, we get $\theta_2 \geqslant 1 + \varphi_2$, and so $\theta_2 = 1 + \varphi_2$.

(vi) Proof of $\theta_1 = 1 + \varphi_1$ when $\alpha(x) \in \bar{\Omega}$ is similar to part (v) above.

Theorem follows from parts (i) to (vi) above.

THEOREM 4. *Let $f(z)$, defined by* (1.5), *be an entire function having generalised order $\rho(\alpha, \alpha, f) \equiv \rho (1 \leqslant \rho \leqslant \infty)$. Then*

(A) (i) *We have*

$$\rho(\alpha, \alpha, f) = P(\tilde{L}),$$

*where*

$$\tilde{L} = \limsup_{n \to \infty} \frac{\alpha(\lambda_n)}{\alpha\{(1/\lambda_n) \log |a_n|^{-1}\}}. \tag{3.7}$$

(ii) $\rho(\alpha, \alpha, f) \leqslant P(\tilde{L}^*)$

*if $\tilde{L}^*$ is well defined by*

$$\tilde{L}^* = \limsup_{n \to \infty} \frac{\alpha(\lambda_n)}{\alpha\{(1/(\lambda_n - \lambda_{n-1})) \log |a_{n-1}/a_n|\}}.$$

(B) *If, further, $\psi(n)$ is ultimately a nondecreasing function then*

$$\rho(\alpha, \alpha, f) = P(\tilde{L}) = P(\tilde{L}^*).$$

*Proof.* (A)(i) Abbreviate $\rho(\alpha, \alpha, f)$ as $\rho$. If $\alpha(x) \in \Omega$, it follows from Theorem 1 of [9] that $\rho \geqslant P(\tilde{L})$.

Next, let $\alpha(x) \in \bar{\Omega}$ and $\rho < \infty$. Then, by (1.6), given $\varepsilon > 0$ there exists $r_0 = r_0(\varepsilon)$ such that

$$\log M(r) < G[\log r; \bar{\rho}] \tag{3.8}$$

for $r > r_0$, $\bar{\rho} = \rho + \varepsilon$. Choose $r = r(n)$ to be the unique root of the equation

$$\lambda_n = \frac{\bar{\rho}}{\log r} G[\log r; \bar{\rho}]. \tag{3.9}$$

Using Cauchy's inequality, (3.8) and (3.9) we get

$$\alpha(\log r) < \alpha \left\{ \frac{\bar{\rho}}{\bar{\rho} - 1} \frac{1}{\lambda_n} \log |a_n|^{-1} \right\}, \tag{3.10}$$

where $r = r(n)$ is given by (3.9). Since $\alpha(x) \in \bar{\Omega}$, as $n \to \infty$, we have

$$\alpha(\log r) \sim \frac{1}{\bar{\rho} - 1} \alpha(\lambda_n) \tag{3.11}$$

for $r = r(n)$ satisfying (3.9). Thus (3.10) and (3.11) give $\rho \geqslant 1 + \tilde{L}$.

We proved above $\rho \geqslant P(\tilde{L})$. To prove $\rho \leqslant P(\tilde{L})$ assume that $\tilde{L} < \infty$. Then by (3.7), given $\varepsilon > 0$, there exists $n'_0 = n'_0(\varepsilon)$ such that

$$|a_n| < \exp \left\{ -\lambda_n G\left[\lambda_n; \frac{1}{\bar{L}}\right] \right\}$$

for $n > n'_0$, where $\bar{L} = \tilde{L} + \varepsilon$. Now,

$$\begin{aligned} M(r) &\leqslant \sum_{n=0}^{\infty} |a_n| r^{\lambda_n} \\ &= \sum_{n=0}^{n'_0} |a_n| r^{\lambda_n} + \sum_{n=n'_0+1}^{s} |a_r| r^{\lambda_n} + \sum_{n=s+1}^{\infty} |a_n| r^{\lambda_n}, \end{aligned}$$

where $s$ is chosen such that

$$\lambda_s \leqslant G[\log 2r; \bar{L}] < \lambda_{s+1}$$

and so

$$M(r) \leqslant A(n_0') + \exp\{G[\log 2r; \bar{L}] \log r\} \times \sum_{n=0}^{\infty} \exp\left\{-\lambda_n G\left[\lambda_n; \frac{1}{\bar{L}}\right]\right\} + \sum_{n=0}^{\infty} 2^{-n},$$

where $A(n_0')$ is a polynomial of degree at most $\lambda_{n_0'}$. Since both the series in the above expression are convergent, we have, for large values of $r$,

$$(1 + o(1)) \log M(r) \leqslant G[\log 2r; \bar{L}] \cdot \log r. \tag{3.12}$$

Using (3.12) we get $\rho \leqslant P(\bar{L})$ and so $\rho = P(\bar{L})$.

This proves part (A)(i) of the theorem.

Parts (A)(ii) and (B) of the theorem can be proved by suitably modifying the proof of Theorem 3 of [4].

LEMMA 1. *Let* (1.5) *be an entire function having generalised lower order* $\lambda(\alpha, \alpha, f) = \lambda (1 \leqslant \lambda \leqslant \infty)$. *Let* $\{n_k\}_{k=0}^{\infty}$ *be an increasing sequence of positive integers. Then*

$$\text{(i)} \qquad \lambda(\alpha, \alpha, f) \geqslant P_\chi(\tilde{l}), \tag{3.13}$$

*where*

$$\chi \equiv \chi(\{n_k\}) = \liminf_{k\to\infty} \frac{\alpha(\lambda_{n_{k-1}})}{\alpha(\lambda_{n_k})} \tag{3.14}$$

*and*

$$\tilde{l} \equiv \tilde{l}(\{n_k\}) = \liminf_{k\to\infty} \frac{\alpha(\lambda_{n_{k-1}})}{\alpha\{(1/\lambda_{n_k}) \log |a_{n_k}|^{-1}\}}. \tag{3.15}$$

(ii) *Further, if we take* $\alpha(x) = \alpha(a)$ *on* $(-\infty, a)$, *then*

$$\lambda(\alpha, \alpha, f) \geqslant P_\chi(\tilde{l}^*), \tag{3.16}$$

*where* $\chi$ *is as in* (3.14) *and*

$$\tilde{l}^* \equiv \tilde{l}^*(\{n_k\}) = \liminf_{k\to\infty} \frac{\alpha(\lambda_{n_{k-1}})}{\alpha\{(1/(\lambda_{n_k} - \lambda_{n_{k-1}})) \log |a_{n_{k-1}}/a_{n_k}|\}}. \tag{3.17}$$

*Proof.* Abbreviate $\lambda(\alpha, \alpha, f)$ as $\lambda$. If $\alpha(x) \in \Omega$, it follows from part (iii) of the proof of Theorem 4 of Shah [10] that $\lambda \geqslant P(\tilde{l})$.

Next, let $\alpha(x) \in \bar{\Omega}$ and $\tilde{l} < \infty$. Then, for $k > k_0(\varepsilon)$, $\varepsilon > 0$, we have

$$|a_{n_k}| > \exp\{-\lambda_{n_k} G[\lambda_{n_{k-1}}; 1/\bar{l}]\}, \qquad \bar{l} = \tilde{l} = \varepsilon. \tag{3.18}$$

Set $r_k = \exp\{2G[\lambda_{n_{k-1}}; 1/\bar{l}]\}$. Then, for $r_k \leqslant r \leqslant r_{k+1}$, we have, by Caychy's inequality and (3.18), that

$$\log M(r) \geqslant \lambda_{n_k} G[\lambda_{n_{k-1}}; 1/\bar{l}]. \tag{3.19}$$

Using (3.19), we get

$$\frac{\alpha(\log M(r))}{\alpha(\log r/2)} \geqslant \frac{\alpha(\lambda_{n_k} G[\lambda_{n_{k-1}}; 1/\bar{l}])}{\alpha(\log r_{k+1}/2)}. \tag{3.20}$$

Equation (3.13) now follows from (3.20) for $\bar{l} < \infty$. For $\bar{l} = \infty$ we get $\lambda = \infty$.

This proves part (i) of the lemma.

(ii) One can verify that

$$\bar{l}(\{n_k\}) \geqslant \bar{l}^*(\{n_k\}) \tag{3.21}$$

for any increasing sequences $\{n_k\}$ of positive integers, if we take $\alpha(x) = \alpha(a)$ on $(-\infty, a)$. Part (ii) of the lemma now follows from (3.13) and (3.21).

This proves the lemma.

LEMMA 2. *Let* (1.5) *be an entire function with* $\psi(n)$ *ultimately a nondecreasing function of n. Then*

$$l_0 = \liminf_{n \to \infty} \frac{\alpha(\lambda_{n-1})}{\{(1/\lambda_n) \log |a_n|^{-1}\}} \geqslant \varphi_2; \tag{3.22}$$

*also*

$$l_0^* = \liminf_{n \to \infty} \frac{\alpha(\lambda_{n-1})}{\alpha\{(1/(\lambda_n - \lambda_{n-1})) \log |a_{n-1}/a_n|\}} \geqslant \varphi_2, \tag{3.23}$$

*where* $\varphi_2$, *for* $\alpha(x) \in \Lambda$, *is defined by* (3.3).

Proof of the lemma can be constructed, with suitable changes, along the lines of proofs of Lemma 4 of [4] and Theorem 2(ii) of [10].

In view of Theorem 3, Lemma 1 and Lemma 2, we have

THEOREM 5. *Let* (1.5) *be an entire function with generalised lower order* $\lambda(\alpha, \alpha, \lambda) \equiv \lambda (1 \leqslant \lambda \leqslant \infty)$, *and* $\psi(n)$ *be ultimately a nondecreasing function of n. Then,*

(i) *If* $\alpha(x) \in \Omega$, *we have*

$$\lambda = P(l_0) = P(l_0^*), \tag{3.24}$$

*where* $l_0$ *and* $l_0^*$ *are defined by* (3.22) *and* (3.23), *respectively.*

(ii) *If* $\alpha(x) \in \bar{\Omega}$, *then* (3.24) *holds under the additional condition that* $\alpha(\lambda_n) \sim \alpha(\lambda_{n+1})$ *as* $n \to \infty$.

We now prove

THEOREM 6. *Let* (1.5) *be an entire function with generalised lower order* $\lambda(\alpha, \alpha, f) \equiv \lambda (1 \leqslant \lambda \leqslant \infty)$. *Then*

(i) *If* $\alpha(x) \in \Omega$, *we have*

$$\lambda = \max_{\{n_k\}} [P_\chi(\tilde{l})] \tag{3.25}$$

*and, further, if we take* $\alpha(x) = \alpha(a)$ *on* $(-\infty, a)$, *then*

$$\lambda(\alpha, \alpha, f) = \max_{\{n_k\}} [P_\chi(\tilde{l}^*)], \tag{3.26}$$

*where* $\chi$, $\tilde{l}$ *and* $\tilde{l}^*$ *are as defined in* (3.14), (3.15) *and* (3.17), *respectively, and maximum is taken over all increasing sequences* $\{n_k\}$ *of positive integers.*

(ii) *Further, if* $\{\lambda_{n_m}\}$ *is the sequence of the principal indices of* $f(z)$ *such that* $\alpha(\lambda_{n_m}) \sim \alpha(\lambda_{n_{m+1}})$ *as* $m \to \infty$, *then* (3.25) *and* (3.26) *hold for* $\alpha(x) \in \bar{\Omega}$ *also (here again, in* (3.26) *we take* $\alpha(x) = \alpha(a)$ *on* $(-\infty, a)$).

*Proof.* Consider the function $\bar{g}(z) = \sum_{m=0}^{\infty} a_{n_m} z^{\lambda_{n_m}}$, where $\{\lambda_{n_m}\}$ is the sequence of the principal indices of $f(z)$. Then $\bar{g}(z)$ is also an entire function. Further, $f(z)$ and $\bar{g}(z)$ have the same maximum term for any $z$, and so by Theorem 3 the generalised lower order of $\bar{g}(z)$ is also $\lambda(\alpha, \alpha, f)$. Also, $\bar{g}(z)$ satisfies the hypothesis of Theorem 5. Now, applying Theorem 5 to $\bar{g}(z)$ and Lemma 1 to $f(z)$, we get (3.25) and (3.26). This proves the theorem.

*Remark.* With $\alpha(x) = l_p x$, $p \geqslant 1$, we get many results of [4, 11], from the results of this section. Here $l_j x$ denotes the $j$th iterate of $\log x$.

## 4. PROOFS OF THEOREMS 1 AND 2

We first have the following connecting lemma:

LEMMA 3. *Let* $f(x) \in C[-1, 1]$ *and* $E_n(f)$ *satisfy* (1.2). *Then* (A) *of Theorem* 1 *holds. Further* $g(z) = \sum_{n=0}^{\infty} E_n(f) z^n$ *is an entire transcendental function and*

$$\rho(\alpha, \alpha, f) = \rho(\alpha, \alpha, g)$$

*and*

$$\lambda(\alpha, \alpha, f) = \lambda(\alpha, \alpha, g).$$

The lemma follows from well known inequalities [5, pp. 76–78] and we omit the proof.

*Proofs of Theorems* 1 *and* 2. Theorems 1 and 2 now follow easily from the Theorems 4–6 and Lemmas 1–3.

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## REFERENCES


1. S. BERNSTEIN, "Leçons sur les propriétés extrémales et al meilleure approximation des fonctions analytiques d'une variable réele," Gauthier–Villars, Paris, 1926.
2. R. P. BOAS, JR., "Entire functions," Academic Press, New York, 1954.
3. O. P. JUNEJA, Approximation of an entire function, *J. Approx. Theory* **11** (1974), 343–349.
4. O. P. JUNEJA, G. P. KAPOOR, AND S. K. BAJPAI, On the $(p, q)$-order and lower $(p, q)$-order of an entire function, *J. Reine Angew. Math.* **282** (1976), 53–67.
5. G. G. LORENTZ, "Approximation of Functions," Holt, Rinehart & Winston, New York, 1966.
6. G. MEINARDUS, "Approximation of functions: Theory and Numerical Methods," Springer-Verlag, New York/Berlin, 1967.
7. A. R. REDDY, Approximation of an entire function, *J. Approx. Theory* **3** (1970), 128–137.
8. A. R. REDDY, Best polynomial approximation to certain entire functions, *J. Approx. Theory* **5** (1972), 97–112.
9. M. N. ŠEREMETA, On the connection between the growth of the maximum modulus of an entire function and the moduli of the coefficients of its power series expansion, *Amer. Math. Soc. Transl. (2)* **88** (1970), 291–301.
10. S. M. SHAH, Polynomial approximation of an entire function and generalised orders, *J. Approx. Theory* **19** (1977), 315–324.
11. S. M. SHAH AND M. ISHAQ, On the maximum modulus and the coefficients of an entire series, *J. Indian Math. Soc.* **16** (1952), 177–182.
12. G. VALIRON, "Lectures on the General Theory of Integral Functions," Chelsea, New York, 1949.
13. R. S. VARGA, On an extension of a result of S. N. Bernstein, *J. Approx. Theory* **1** (1968), 176–179.